Nonlinear Strain Analysis of Carbon Fibre Reinforced Polymer Laminates

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Abstract: An approach describing the nonlinear behavior of multidirectional carbon fibre reinforced polymer composites has been elaborated as application of matrix algorithms to classical lamination theory relations. It is assumed that nonlinear in-plane shear strain arising in each lamina plays a dominant role in such nonlinear behavior. Analytical approximation of the lamina shear stress-strain curve is used and substituted into the lamina stiffness matrix. It allows the stiffness matrix to be divided into two constituents describing consequently linear and nonlinear properties of the lamina. To obtain the stiffness matrix of the multidirectional laminate, the classical lamination theory relations are used. Matrix algorithms are applied for the purpose of performing the inverse of the stiffness matrix resulting in the explicit form of the compliance matrix. The engineering characteristics of the laminate may be obtained by using this approach. The stress-strain curves predicted for some lay ups are compared with the experimental data and a satisfactory agreement is observed.

Key Words: classical lamination theory, matrix algorithms, angle-ply laminates

1. Introduction

Predicting the mechanical behavior of composite materials is very important because of their widespread use in aerospace, machine-building, automotive and other industries. Intensive experimental and theoretical investigations in the field of strength and failure of composite materials were conducted during the last decades. First of all, WWFE-World-Wide Failure Exercise (Hinton et al. (eds.), 2004) should be noted. It was based on numerous papers of researchers from various countries. Most of the WWFE papers consider the unidirectional layer as a base for predicting the laminate properties, as well as the laminate non-elastic properties as a result of the nonlinear properties of the layer. As the nonlinear properties of the layer are induced by nonlinear properties under in-plane shear of the layer, it is important to get the shear stress-strain curve. There are different kinds of approximation of such curves, but the principal idea of the approach proposed in this paper is to separate of elastic strain from the non-elastic one.

As to the elasticity range, it is important to estimate correctly the lamina mechanical properties. This problem may be successfully solved with the aid of the identification method (Dumansky et al., 2011). The method is based on minimization of residual between experimental and calculated data. Testing of various angle-ply laminates is preferable to define experimentally the lamina properties (Hinton et al., 1996; Bogetti et al., 2004; Dumansky et al., 2011, Alimov et al., 2012). It should be noted that significant nonlinearity of stress-strain curves is observed in case of loading not coinciding considerably with the fibre direction of any of the layers. In some cases (Bogetti et al., 2004; Zinoviev et al., 2004) such nonlinearity is taken into account by applying the incremental algorithms to stresses or strains. It is possible to use analytical approximation of the stress-strain curves and, in particular, to apply the Ramberg-Osgood nonlinear three-parameter equation (Bogetti et al., 2004). There were some other attempts to describe the lamina nonlinear behavior (Petit and Waddups, 1969; Rosen, 1972; Zinoviev et al., 2004), but elaborating the methods suitable for practical use and providing satisfactory agreement with experimental data is still of current importance.

2. Identification of lamina elastic properties

Using the identification method is preferable for defining the lamina properties of angle-ply specimens in comparison with the method based on testing of unidirectional laminates under off-axis loading. In the general case the problem of identification is reduced to minimization of a residual function providing for, to some extent, the minimization between theoretical and experimental data. In particular, the determination of the lamina properties can be carried out by the use of testing data of angle-ply lay ups, strains being measured in both longitudinal and transverse directions. This can be expressed by the following relation (Dumansky et al., 2011)

$$\min_{g_{ij}^{0}}\left[\sum\left(\varepsilon_x^{calc}-\varepsilon_x^{\exp}\right)^2+\left(\varepsilon_y^{calc}-\varepsilon_y^{\exp}\right)^2\right] \tag{1}$$

where g_{ij}^0 are the lamina stiffness components in the main axes of orthotropy, $\varepsilon_x^{calc}, \varepsilon_y^{calc}$ $\varepsilon_x^{\exp}$, $\varepsilon_y^{\exp}$ are the calculated and experimental strains of angle-ply laminate correspondingly.

In case of uniaxial loading the matrix form of constitutive equations is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ 0 \end{Bmatrix} = \begin{pmatrix} s_{xx} & s_{xy} & 0 \\ s_{xy} & s_{xx} & 0 \\ 0 & 0 & s_{ss} \end{pmatrix} \begin{Bmatrix} \sigma_x \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

where s_{xx} s_{xy}, s_{yy}, s_{ss} are the compliance components of angle-ply laminate.

Since the matrices of compliance and stiffness are connected by the equality $[S_{xy}] = [G_{xy}]^{-1}$ the relations for the strains may be written in the following form:

$$\begin{aligned} \varepsilon_x^{calc} &= \frac{g_{xx}}{D_{xy}} \sigma_x; \\ \varepsilon_y^{calc} &= -\frac{g_{xy}}{D_{xy}} \sigma_x \end{aligned} \tag{3}$$

where $D_{xy} = g_{xx} g_y - g_{xy}^2$.

In its turn the relations of stiffness components of angle-ply laminate $[\pm\theta]$ may be written as (Hinton et al., 1996)

$$\begin{aligned} g_{xx} &= g_{11}^0 \cos^4\theta + g_{22}^0 \sin^4\theta + 2\left(g_{12}^0 + 2g_{66}^0\right)\sin^2\theta\cos^2\theta \\ g_{yy} &= g_{11}^0 \sin^4\theta + g_{22}^0 \cos^4\theta + 2\left(g_{12}^0 + 2g_{66}^0\right)\sin^2\theta\cos^2\theta \\ g_{yy} &= \left(g_{11}^0 + g_{22}^0 - 4g_{66}^0\right)\sin^2\theta\cos^2\theta + \left(\sin^4\theta + \cos^4\theta\right)g_{12}^0, \end{aligned} \tag{4}$$

where $g_{11}^0 = \dfrac{E_1}{1-\nu_{12}\nu_{21}}$, $g_{22}^0 = \dfrac{E_2}{1-\nu_{12}\nu_{21}}$, $g_{12}^0 = \dfrac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}$, $g_{66}^0 = G_{12}$; E_1, E_2, G_{12} and ν_{12} are the lamina engineering constants.

Thus Equations (1) to (4) allow all the lamina elastic properties to be defined.

3. Non-linear shear stress-strain curve approximation

The experimental results allow us to assume that for the lamina elastic moduli in the main orthotropy axes E_1, E_2 and Poisson's ratio ν_{12} may be taken invariant. The reason of nonlinear behavior is the lamina in-plane shear properties. The shear stress-strain curves may be given in advance as in the WWFE instruction or defined from the indirect experimental data. These curves may be approximated by a function (for example by piecewise linear one). A diagram in the piecewise form is shown in Fig. 1.

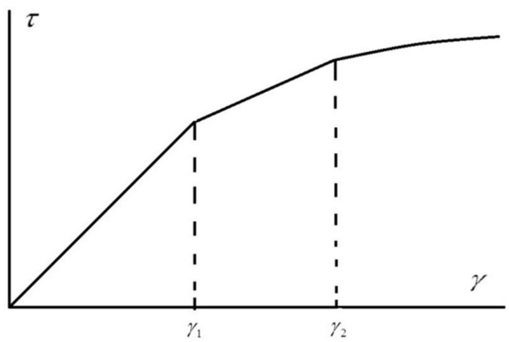

Figure 1. Lamina in-plane shear stress-strain approximation by piecewise function.

Such an approximation may be analytically expressed as follows

$$g_{66}(\gamma) = g_{66}^{0} - \Delta g_{66}^{(1)} H(\gamma - \gamma_1) - \Delta g_{66}^{(2)} H(\gamma - \gamma_2)\ldots \tag{5}$$

or in case of approximation of the nonlinear area by power function $\tau(\gamma) = \tau(\gamma_1) + g(\gamma - \gamma_1)^n$

$$g_{66}(\gamma) = g_{66}^{0} - gn(\gamma - \gamma_1)^{n-1} H(\gamma - \gamma_1) \tag{6}$$

where g_{66}^{0} is shear modulus of the linear area; $\Delta g_{66}^{(k)}$ characterizes change in shear modulus of the r-*th* part of the strain axis; $H(\ )$ is Heaviside unit function; g and n are parameters of the power function.

It should be noted for (5) that the material reaches the yield area when $\sum_{k=1}^{n} \Delta g_{66}^{(k)} = g_{66}^{0}$. Experimental shear stress-strain curves for the materials given to contributors of the Third World-Wide Failure Exercise (WWFE-III) Part (A) (Kaddour et al., 2011) are shown in Fig. 2.

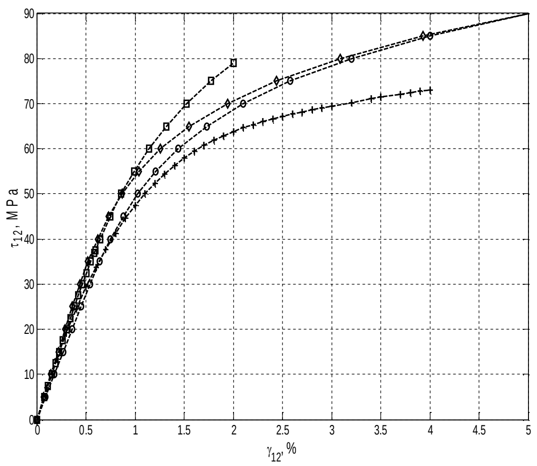

Figure 2. Longitudinal lamina shear stress-strain curves. Glass/LY556 (crosses), carbon/IM7/8552 (circles), G40-800/5260 (diamonds), AS4/3501-6 (squares).

As is seen from Fig.2 the lamina shear-stress strain curves are essentially nonlinear and it is possible to represent them by (5) or (6). Two or three linear parts may be sufficient to describe the lamina shear stress-strain curve (Dumansky and Tairova, 2008). In the general case, the approximation by (5) and (6) may be rewritten in the following form

$$g_{66}(\gamma) = g_{66}^{0} - f(\gamma) \tag{7}$$

Then the lamina stiffness matrix may be expressed as

$$[G_{12}] = \begin{pmatrix} g_{11}^{0} & g_{12}^{0} & 0 \\ g_{12}^{0} & g_{22}^{0} & 0 \\ 0 & 0 & g_{66}^{0} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} f \tag{8}$$

For the purpose of matrix calculus equation (8) may be represented as

$$[G_{12}] = (G_{12}^{0}) - [I_0] f\ . \tag{9}$$

Matrix stiffness of the laminate is formed in accordance with classical lamination theory relations (Rabotnov, 1979)

$$\left[G_{xy}\right]=\sum_{k}\left[T_{1}^{(k)}\right]\left[G_{12}\right]\left[T_{1}^{(k)}\right]^{T}\bar{h}^{(k)} \tag{10}$$

where

$$\left[T_{1}^{(k)}\right]=\begin{pmatrix} c_{(k)}^{2} & s_{(k)}^{2} & -2s_{(k)}c_{(k)} \\ s_{(k)}^{2} & c_{(k)}^{2} & 2s_{(k)}c_{(k)} \\ s_{(k)}c_{(k)} & -s_{(k)}c_{(k)} & c_{(k)}^{2}-c_{(k)}^{2} \end{pmatrix} \tag{11}$$

is transformation matrix; $s_{(k)}=\sin\theta_k$; $c_{(k)}=\cos\theta_k$; θ_k is angle between axis Ox and the fibre direction of the k -th lamina.

Substituting the stiffness matrix of (9) into (10) yields

$$\left[G_{xy}\right]=\left[G_{xy}^{0}\right]-\left[\tilde{G}_{xy}\right]f \tag{12}$$

where $\left[G_{xy}^{0}\right]=\sum_{k}\left[T_{1}^{(k)}\right]\left[G_{12}^{0}\right]\left[T_{1}^{(k)}\right]^{T}\bar{h}^{(k)}$, $\left[\tilde{G}_{xy}\right]=\sum_{k}\left[T_{1}^{(k)}\right]\left[I_{0}\right]\left[T_{1}^{(k)}\right]^{T}\bar{h}^{(k)}$. Thus, the explicit form of the stiffness matrix has been obtained.

It is necessary also to obtain the compliance matrix. The matrix of compliance is the stiffness matrix inverse. The compliance matrix may be presented as follows (Dumansky et al., 2011)

$$\left[S_{xy}\right]=\left[R\right]diag\left(\frac{1}{1-\lambda_{i}f}\right)\left[R\right]^{-1}\left[S_{xy}^{0}\right] \tag{13}$$

where λ_i are the eigenvalues of matrix $\left[G_{xy}^{0}\right]^{-1}\left[\tilde{G}_{xy}\right]$; matrix $\left[R\right]$ is a result of matrix diagonalisation: $\left[G_{xy}^{0}\right]^{-1}\left[\tilde{G}_{xy}\right]=\left[R\right]diag\left(\lambda_{1}\quad\lambda_{2}\quad\lambda_{3}\right)\left[R\right]^{-1}$. With the aid of (13) the engineering constants of the laminate may be obtained (Dumansky et al., 2011) and the compliance matrix takes the following form:

$$\left[S_{xy}\right]=\begin{pmatrix} \dfrac{1}{E_x} & -\dfrac{\nu_{yx}}{E_y} & 0 \\ -\dfrac{\nu_{xy}}{E_x} & \dfrac{1}{E_y} & 0 \\ 0 & 0 & \dfrac{1}{G_{xy}} \end{pmatrix} \tag{14}$$

The eigenvalues are defined by lay-up of the laminate; function f characterizes the in-plane shear properties degradation.

4. Experimental procedure

To illustrate the approach, flat specimens $\left[0\right]_4$, $\left[\pm20\right]_2$, $\left[\pm40\right]_2$, $\left[\pm50\right]_2$, $\left[\pm70\right]_2$, $\left[90\right]_4$, lay-ups were tested under quasi-static tensile loading. Minimizing (1) within the elastic strain area the lamina engineering constants were defined. The most of strain nonlinearity is observed for $\left[\pm40\right]_2$, $\left[\pm50\right]_2$ lay-ups which is induced by significant in-plane shear strain of the lamina. Longitudinal and transverse stress-strain curves are presented in Fig. 3 and 4.

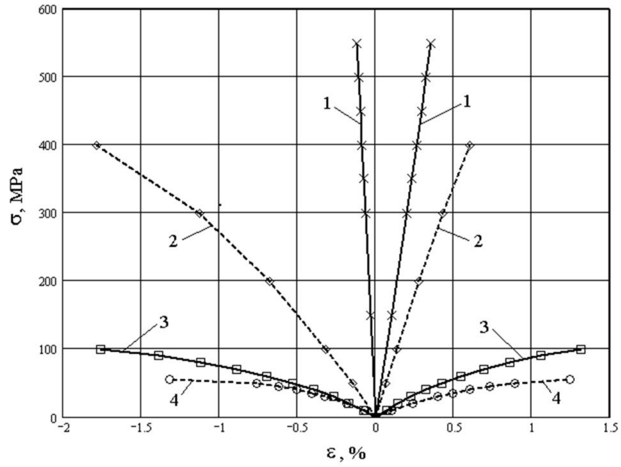

Figure 3. Longitudinal and transverse stress-strain curves for 1 - $[0]_4$; 2 - $[\pm 20]_2$ 3 - $[\pm 40]_2$; 4 - $[\pm 50]_2$ lay-ups.

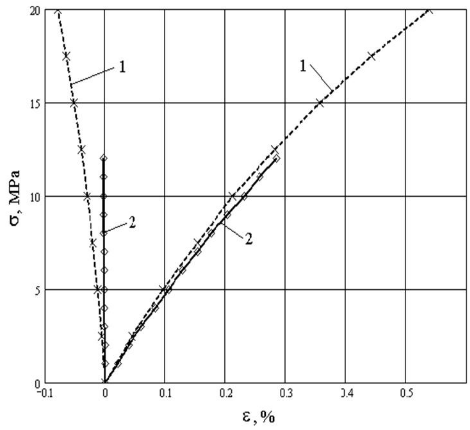

Figure 4. Longitudinal and transverse stress-strain curves for 1 - $[\pm 70]_2$ 2 - $[90]_4$ lay-ups.

5. Model of Application

Elastic constants defined by means of the identification method are as follows: $E_1 = 184$ GPa, $E_2 = 11.1$ GPa, $E_2 = 6.5$ GPa, $\nu_{12} = 0.33$. Nonlinear in-plane shear of the lamina is defined by the following equation:

$$\tau(\gamma) = g_{66}^0 \gamma - \left[g_{66}^0 (\gamma - \gamma_*) - g (\gamma - \gamma_*)^n \right] H(\gamma - \gamma_*) \tag{15}$$

where $\gamma_* = 0.7$ % is the threshold value of shear strain corresponding to the beginning of nonlinearity; g and n are the parameters of the power function approximating the nonlinear strain. They are as follows: $g = 2.2$ GPa, $n = 0.95$. Then the shear modulus approximation may be written as

$$g_{66} = g_{66}^0 - \left[g_{66}^0 - gn(\gamma - \gamma_*)^{n-1} \right] H(\gamma - \gamma_*) \tag{16}$$

In case of linear approximation the shear modulus is stated as

$$g_{66} = g_{66}^0 - \Delta g H(\gamma - \gamma_*) \tag{17}$$

where $\Delta g \approx 5.4$ GPa.

Using the compliance matrix of (13) the longitudinal and transverse strains may be calculated as

$$\begin{aligned}\varepsilon_x^{(i+1)} &= \varepsilon_x^{(i)} + s_{xx}^{(i)}\left(\sigma_x^{(i+1)} - \sigma_x^{(i)}\right)\\ \varepsilon_y^{(i+1)} &= \varepsilon_y^{(i)} + s_{xy}^{(i)}\left(\sigma_x^{(i+1)} - \sigma_x^{(i)}\right)\end{aligned} \tag{18}$$

Comparison of experimental and calculated data is shown in Fig. 5.

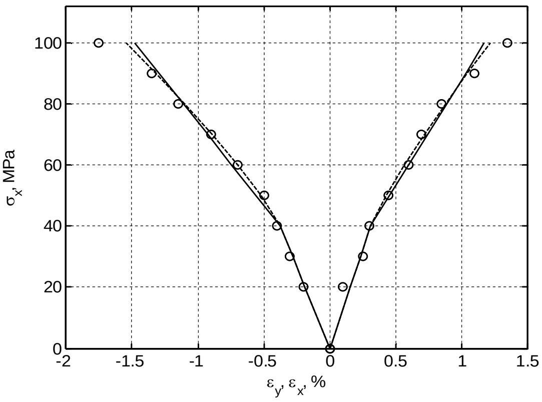

Figure 5. Comparison of the experimental data (circles) and calculated stress-strain curves for $[\pm 40]_2$ lay-up. Solid line is calculated by using piecewise linear approximation of shear stress-strain curve, dashed line for power one.

6. Conclusion and recommendations

The analytical model proposed on the basis of matrix algorithms and classical lamination theory satisfactorily describes the nonlinear stress-strain response of carbon fiber reinforced laminates.

Applying the matrix algorithms to the classical lamination theory relations has proved to be highly effective in obtaining the matrices of stiffness and compliance of the laminates that allow the nonlinear mechanical properties to be described.

The approach may be generalized and extended to prediction of the rheological properties of carbon fiber reinforced plastics, as well as applied for describing the behavior of laminates under the time variable loadings. On the basis of algebra of resolvent operators (Rabotnov, 1979) a matrix method for the construction of hereditary constitutive equations was elaborated (Dumansky et al., 2008). Combining such a matrix method with the approach proposed in this paper allows the anisotropy of nonlinear and rheological properties to be described.

The approach proposed may be used in strength analysis and design of thin-walled structures of composite laminates.

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