Equation Chapter 1 Section 1

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Construction of Hereditary Constitutive equations of Composite Laminates

A.M. Dumansky [1]*, L.P. Tairova [2]

[1] *Laboratory Methods of Fatigue Certifying Tests of Materials, Mechanical Engineering Research Institute of Russian Academy of Sciences, Maly Kharitonievsky, 4, Moscow, 101990, Russia*
Email: alduman@rambler.ru
[2] *Bauman Moscow State Technical University, 2nd Baumanskaya, 5, Moscow, 105005, Russia*
Email: com-or@yandex.ru

Abstract The analytical method of construction of constitutive equations for composite laminates based on algebra of resolvent operators and relations of the classical lamination theory was suggested. On the basis of hereditary constitutive equations of the unidirectional layer with elastic behavior in the main directions of the orthotropy and viscoelastic behavior under shear the constitutive hereditary equations of the laminate can be obtained. The constitutive equations can be expressed by means of stiffness, compliance or technical elasticity hereditary operators. The example of construction of the constitutive equations was demonstrated on the cross-ply carbon reinforced laminates.

Key words: hereditary operator, resolvent operator, viscoelastic behavior, cross-ply laminate, matrix resolvent

INTRODUCTION

Viscoelastic materials can be characterized by a number of interrelation functions such as compliance, relaxation and complex moduli [1-3]. Fiber reinforced laminates show evidence of rheological properties to a large extent depending on the polymeric matrix properties. It means that rheological properties of the laminate are defined by shear properties of the layers. Use of relationships of the classical laminate theory to predict viscoelastic properties of the laminates were considered [3-6]. Algebra of resolvent operators and its application to composite materials are elaborated [1]. Cross-ply composites show the most noticeable viscoelastic properties especially for lay-ups close to 45 degrees, in this case maximum shear stresses take place [4,5,7,8].

ANALYTICAL MODEL

The system of the constitutive equations for the unidirectional lamina characterized in matrix form by the only hereditary operator may be presented as [9]

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_1 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{66}^* \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{\nu_{21}}{E_2} & 0 \\ \frac{\nu_{21}}{E_2} & \frac{1}{E_1} & 0 \\ 0 & 0 & \frac{1}{G_{12}^0}\left(1+K^*\right) \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}, \tag{1}$$

where G_{12}^0 is instantaneous shear modulus [10], K^* is a hereditary operator acting on stress as: $K^*\sigma = \int_0^t K\left(t-\tau\right)\sigma\left(\tau\right)d\tau$ and $K\left(t\right)$ is a kernel of the operator.

The inversion of (1) gives

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & g_{66}^* \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}. \tag{2}$$

Relation for stiffness under shear can be obtained with the aid of resolvent of operator K^* [1]:

$$g_{66}^* = \left[\frac{1}{G_{12}^0}\left(1+K^*\right)\right]^{-1} = G_{12}^0\left(1-R^*\right). \tag{3}$$

Matrix of the stiffness for a layer rotated through θ degrees to abscissa axis is

$$\left[G_{xy}^{(\theta)}\right] = [T_\theta][G_{12}][T_\theta]^T, \tag{4}$$

where $[T_\theta]$ is a transition matrix.

Using the classical laminate theory relationships with the help of relation (2) we can obtain global stiffness matrix of the laminate

$$\left[G_{xy}\right] = \sum_{k=1}^{n}\left[G_{xy}^{(k)}\right]\overline{h}^{(k)} \tag{5}$$

After transformations the global stiffness matrix (5) takes the form:

$$\left[G_{xy}\right] = \left[G_{xy}^0\right] - \left[G_{xy}^t\right]R^*. \tag{6}$$

If resolvent operator R^* depends on the parameter μ, with the aid of matrix resolvent the stiffness matrix can be inversed to the following relationship [9]

$$\left[S_{xy}\right] = \left[I + Q\,diag\left(\lambda_i R^*\left(\mu-\beta_i\right)\right)Q^{-1}\right]\left[S_{xy}^0\right], \tag{7}$$

where $\left[S_{xy}\right]$ is the laminate compliance matrix, $[I]$ is a unit matrix and $[Q]$ is the auxiliary matrix calculated by diagonalization of the matrix multiplication $\left[G_{xy}^0\right]^{-1}\left[G_{xy}^t\right] = [Q][D][Q]^{-1}$, $[D] = diag\left(\beta_i\right)$ and β_i are the eigenvalues.

Expressions for the technical characteristics of hereditary elasticity can be obtained with the help of the relationships of the algebra of the resolvent operators [1]

$$\left(1+\beta R^*\left(\mu\right)\right)^{-1} = 1-\beta R^*\left(\mu-\beta\right); \qquad R^*\left(\mu_1\right)R^*\left(\mu_2\right) = \frac{1}{\mu_1-\mu_2}\left(R^*\left(\mu_1\right)-R^*\left(\mu_2\right)\right) \tag{8}$$

The modulus of the laminate can be obtained by inverse of the corresponding component of the compliance matrix

$$E_x = \frac{1}{s_{xx}^0\left(1+\beta_x R^*\left(\mu_x\right)\right)} = E_x^0\left(1-\beta_x R^*\left(\mu_x-\beta_x\right)\right), \tag{9}$$

where $E_x^0 = \frac{1}{s_{xx}^0}$ is instantaneous modulus of elasticity.

The general formulation for Poisson's ratio of the laminate can be derived by consequent use of relations (8) and is given in the next form

$$\nu_{xy} = \frac{g_{xy}}{g_{yy}} = \frac{g_{xy}^0\left(1-\kappa_{xy}R^*\left(\eta_{xy}\right)\right)}{g_{yy}^0\left(1-\kappa_y R^*\left(\eta_y\right)\right)} =$$
$$\nu_{xy}^0\left[1-\left(\kappa_{xy}-\frac{\kappa_y\kappa_{xy}}{\eta_y+\kappa_y-\eta_{xy}}\right)R^*\left(\eta_{xy}\right)+\left(\kappa_y-\frac{\kappa_y\kappa_{xy}}{\eta_y+\kappa_y-\eta_{xy}}\right)R^*\left(\eta_y+\kappa_y\right)\right]. \tag{10}$$

The previously derived relations for technical hereditary characteristics and the other ones can be used for describing long term properties of the laminates under various kinds of loading. The form of the constitutive equations will depend on the laminate lay-up and kind of the loading.

RESULTS AND VERIFICATION OF THE MODEL

The method of the construction of the constitutive equations was demonstrated on example of cross-ply carbon fiber reinforced plastic. The moduli of elasticity and Poisson's ratio were determined by identification method [11] on quasistatic tension data of flat specimens of $[0]_4$, $[\pm10]_2$, $[\pm20]_2$, $[\pm40]_2$, $[\pm50]_2$, $[\pm70]_2$, $[90]_4$ lay-ups. Since elasticity in the main axes of the layer orthotropy - moduli E_1, E_2 and Poisson's ratio ν_{12} are the instantaneous characteristics of the layer. Elasticity characteristics of the unidirectional layer turn out to be equal to: $E_1 = 150$ GPa, $E_2 = 3.95$ GPa, $\nu_{12} = 0.315$. Shear modulus $G_{12} = 2.39$ GPa and needs to be corrected to take into account the rheological properties.

Creep is traditionally described by power, exponential or linear combination of exponential functions [1,2,5,8]. Abel's operator belongs to power function for creep decribing. It was established [1] that Rabotnov fraction exponential function is the resolvent of Abel's operator. Power and exponential functions are special cases of Rabotnov fraction exponential function [1].

For describing viscoelastic properties of the unidirectional layer under shear in (1) Abel's operator was chosen: $K^* = kI_\alpha^*$, the kernel is $I_\alpha\left(t\right) = \frac{t^\alpha}{\Gamma\left(1+\alpha\right)}$, $-1<\alpha<0$. Recalculated value of the shear modulus obtained on long-term test data of $[\pm40]_2$ turned out to be equal to 3.62 GPa and the values of Abel's kernel parameters are the following: $\alpha = -0.9$, $k = 1.0465$ $\text{min}^{-(1+\alpha)}$. Rabotnov fraction exponential function is equal to: $\Omega_\alpha^*\left(-\beta\right)\cdot 1 = t^{1+\alpha}\sum_{n=0}^{\infty}\frac{\left(-1\right)^n\left(\beta t^{1+\alpha}\right)^n}{\Gamma\left[1+\left(1+\alpha\right)\left(1+n\right)\right]}$.

In correspondence with the scheme of the matrix resolvent we can extract the relation for compliance component (7)

$$s_{xx} = 0.05505\left(1+.897\Omega_{-0.9}^*\left(-1.014\right)\right). \tag{11}$$

The expression for long term (relaxation) modulus for $[\pm40]_2$ lay-up we can get using inverse of Rabotnov fraction exponential function (9)

$$E_x = \frac{1}{s_{xx}} = 18.16\left(1+0.897\Omega^*_{-0.9}\left(-1.014\right)\right)^{-1} = 18.16\left(1-0.897\Omega^*_{-0.9}\left(-1.911\right)\right). \tag{12}$$

Use the asymptotical property of Rabotnov fraction exponential function shows that $\Omega^*_\alpha\left(-\beta\right)\to\frac{1}{\beta}$, at $t\to\infty$, then the asymptotical value of the modulus is $E_x^\infty = 9.64$ MPa, hence the change of the modulus is about 47%.

The relation for Poisson's ratio (10) in this case can be written

$$\nu_{xy} = \nu^0_{xy}\left(1+\left(k_y-k_{xy}\right)\Omega^*_\alpha\left(-k+k_y\right)\right). \tag{13}$$

Calculated value of Poisson's ratio at $t=\infty$ is equal to

$$\nu^\infty_{xy} = \nu^0_{xy}\left(1+\frac{k_y-k_{xy}}{k-k_y}\right), \tag{14}$$

changes from 1.11 to 1.12 that is negligibly small.

It is important to note that in absence of time dependent properties the hereditary constitutive equations degenerate into the relationships of anisotropic elasticity.

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