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MACHINE MECHANICS

Specific Features of Dynamic Interactions in Nonlinear Systems

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Abstract—Specific features of nonlinear systems determined by the energy interaction of their coordinates are considered. To assess these features, the potential energy surfaces are considered, on which the system dynamics is presented using the trajectories of motion of an image point. The specific features of these trajectories are determined by the topological characteristics of the potential energy surface in the configuration space (local surface curvatures). Unlike linear systems, nonlinear systems are characterized by the presence on the energy surface of additional extremal curvatures and characteristic points: in addition to elliptic points, parabolic and hyperbolic points appear, which determine the character of geodetic lines near these extrema and points. The specific features of the geodetic lines determine the character of free and forced vibrations in the system, its dynamic features, and the presence of nonlinear effects in the system.

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A generalization of the dynamic properties of linear systems using the geometric characteristics of potential energy surfaces was presented in [1]. In [1], the dynamic properties of linear systems were analyzed using an energy method that consists of studying the topological specific features of the potential energy surfaces (and the constant energy surfaces) of the system in the configuration space. The constant energy surfaces of linear systems are represented by n-dimensional ellipsoids. The ellipsoid dimensionality corresponds to the number of degrees of freedom of the system. The ellipsoid vortices determine the eigenfrequencies, the vibration modes, and their orthogonality. The free vibrations of the system correspond to the lines of the principal curvature of the energy surface plotted in the configuration space.

In this study, we develop geometric generalizations for the invariants of the dynamic matrix expressed in terms of the principal curvature of the energy surface in the configuration space.

Owing to nonlinearity of the system, the geometry of energy surfaces becomes more complex and, as a result, its dynamics also becomes more complex; i.e., different nonlinear effects appear. If the nonlinearity is small, the potential energy surface insignificantly differs from that of a linear system. If the nonlinearity in the system grows, the surface topology and the character of geodetic lines determining the dynamics of the system behavior radically change.

The potential energy of a system at a certain point in the configuration space $\mathbf{X}^*$ may be found by calculating the definite integral

$$P = \int_0^{\mathbf{X}^*}\int_0^{\mathbf{X}^*} \mathbf{C}(\mathbf{X})d\mathbf{X}d\mathbf{X},$$

where $\mathbf{C}(\mathbf{X})$ is the matrix of quasi-elastic coefficients whose elements depend on the elements of the system's displacement vector $\mathbf{X}$.

Let us consider a system consisting of two masses with a soft characteristic of the nonlinearity of the elastic element connecting them (the system is assumed to be symmetric). The characteristic of the quasi-elastic coefficient of the elastic element is set by the expression $C_{12} = C_0(1 - \alpha(X_1 - X_2)^2)$. The factor α determines the type of the nonlinearity and its value. If relative displacements in the system grow, the energy surface deforms, and, if $\alpha > 0$, the connectivity of the system coordinates diminishes.

Figure 1 shows the energy surface for a specific nonlinear system with significant nonlinear characteristics. One can see pronounced valleys and areas of convexity (peaks) and a saddle.

Nonlinear features of a system are usually described by means of singular points in the phase space. On the energy surface, characteristic points correspond to singular points. In the considered example, the sur-

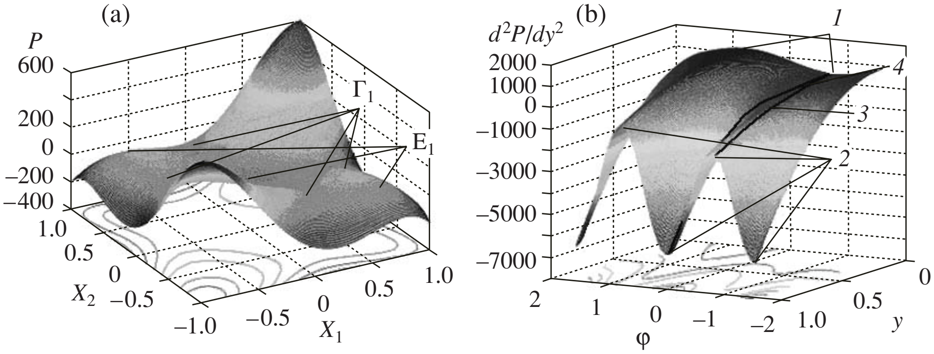

Fig. 1. (a) Energy surface and (b) surface of local curvatures: E_1 denotes elliptic points and Γ_1 denotes hyperbolic points; (*1*) extremal values of principal curvatures (eigenvalues of a linear system; $y \approx 0$); (*2*) extremal values of curvatures of a nonlinear system ($y = y_{\max} = 1, 0$); (*3*) lines of the principal curvature values; (*4*) lines of the additional curvature values.

face contains some characteristic points E_1 and Γ_1 (Fig. 1a). Here, parabolic points may also exist, which are characteristic of a free system (the presence of a zero eigenfrequency). The elliptic point located on a line of the principal curvature of the surface corresponds to singular points on the phase plane of the system with one degree of freedom described by the equation of motion $\ddot{x} + (\omega_0^2 - \alpha x^2)x = 0$. In the case of a system with two degrees of freedom, points Γ_1 with nonzero potential energy ($P \neq 0$) appear, which correspond to saddles in the phase space. In the considered symmetric system, four points Γ_1 are observed, which are equally displaced with respect to the frame of origin (X_1, X_2) and have the energy level P. If the system is not symmetric, the displacements of these points and their energy levels may be different.

While, in the case of a linear system characterized by a constant energy ellipsoid, the dynamic properties are determined by the ordered set of the ellipsoid vortices in the $\bar{n}$-dimensional space, which characterize the frequency spectrum and the modes of free vibrations $L_i(\mathbf{X})$ ($i = 1, \ldots, n$) for $P = 1$ [1], for a nonlinear system, these characteristics are determined by the points $T_i^k(\mathbf{X}, P)$ ($i = 1, \ldots, n, j = 1, \ldots, 3$, where i is the number of points of the jth type) and their spatial distribution with the corresponding values of the energy level. The presence of hyperbolic points on the energy surface results in the appearance of additional extremal curvatures.

The analysis of the number of characteristic points, their distribution by energy levels, and the spatial distribution in the configuration space yields the dynamic properties of the system. Usually, characteristic points of the hyperbolic type correspond to nonzero values of the potential energy ($P > 0$) and elliptic-type points more often correspond to the zero point of the configuration space with $P = 0$ and the properties of the system are close to those of linear systems.

The characteristic points and specific features of energy surfaces may be described in terms of their quadratic form, principal curvatures, Gaussian curvature, and asymptotic lines [2]. To determine the type of characteristic points and their location, one can use the methods of catastrophe theory, for example, the Morse lemma about the *l*-saddle, the Euler characteristic of a surface, the Poincaré theorem, and Betti numbers [3–6]. In catastrophe theory, critical properties of a surface are considered and in this case the characteristic points are called critical points. In the variational calculus, geodetic curves and stationary points corresponding to characteristic points are considered.

We will now study the character of the local curvatures of the energy surface of a nonlinear system in the configuration space for a system with two degrees of freedom. Let us determine the function that describes the change in the curvature depending on the point in the configuration space. We assume that the nonlinearity of elastic elements is described by the function $C = C_0(1 + \alpha x^2)$. Depending on the sign of the nonlinearity coefficient, the nonlinearity characteristic may be hard or soft. The equations of motion have the following form:

$$A_{11}\ddot{X}_1 + C_{11}(1 + \alpha_1 X_1^2)X_1 + C_{12}(1 + \alpha_{12}(X_1 - X_2)^2)X_1 - C_{12}(1 + \alpha_{12}(X_1 - X_2)^2)X_2 = 0,$$

$$A_{22}\ddot{X}_2 + C_{22}(1 + \alpha_2 X_2^2)X_2 + C_{12}(1 + \alpha_{12}(X_1 - X_2)^2)X_2 - C_{12}(1 + \alpha_{12}(X_1 - X_2)^2)X_1 = 0.$$

The potential energy of the system may then be expressed as the following function (the potential function):

$$P = \frac{1}{2}X_1^2\{C_{11}(1-\alpha_1 X_1^2) + C_{12}[1-\alpha_{12}(X_1^2 - 2X_1X_2 + X_2^2)]\}$$
$$+\frac{1}{2}X_2^2\{C_{22}(1-\alpha_1 X_2^2) + C_{12}[1-\alpha_{12}(X_1^2 - 2X_1X_2 + X_2^2)]\}$$
$$-C_{12}X_1X_2(1-\alpha_{12}(X_2^2 - 2X_1X_2 + \alpha_{12}X_1^2)).$$

We perform next a transformation of the coordinates $\mathbf{X} = \mathbf{V}(\varphi)\mathbf{Y}$, where $\mathbf{V}(\varphi)$ is a matrix of transformation to a rotating coordinate frame (φ is the angle of rotation of the coordinate frame). We then find the second derivative of the potential function along the y_1 axis of the rotating coordinate frame

$$\frac{d^2P}{dy_1^2} = C_{11}\cos^2(\varphi) + C_{22}\sin^2(\varphi) + C_{12}\cos^2(\varphi) + C_{12}\sin^2(\varphi) - 2C_{12}\cos(\varphi)\sin(\varphi)$$
$$-6\alpha_{12}C_{12}y_1^2\cos^4(\varphi) - 6\alpha_{12}C_{12}y_1^2\sin^4(\varphi) - 6\alpha_{11}C_{11}y_1^2\cos^4(\varphi) - 6\alpha_{22}C_{22}y_1^2\sin^4(\varphi)$$
$$+24\alpha_{12}C_{12}y_1^2\sin^3(\varphi)\cos(\varphi) + 24\alpha_{12}C_{12}y_1^2\cos^3(\varphi)\sin(\varphi) - 36\alpha_{12}C_{12}y_1^2\cos^2(\varphi)\sin^2(\varphi).$$

The obtained expression determines the dependence of the curvature of the energy surface $K(y_1, \varphi)$ on the direction of the y_1 axis in the configuration space (as a function of φ) and the displacement of a point along this axis. The energy surface for a system with a soft nonlinearity characteristic is shown in Fig. 1a and the surface $K(y_1, \varphi)$ is displayed in Fig. 1b. For small displacements, nonlinearity manifests itself in an insignificant way. In Fig. 1b, one can clearly see the values of the principal curvatures *1*, determining the eigenvalues of the dynamic matrix and illustrating their extremal properties. If displacements grow, additional extremal curvatures of the surface *2* appear (Fig. 1b). Curve *3* shows how the value of the principal curvature changes if displacements in the system grow; i.e., it decreases. Curve *4* indicates the existence of an additional extremal curvature that also decreases if displacements in the system grow. In the case of a system with soft nonlinearity, the curves of the additional extremal curvature pass through the origin of the frame and the hyperbolic points on the potential energy surface. A similar surface may also be plotted for a system with a hard nonlinearity characteristic. In this case, a growth of system displacements results in an increase in the principal and additional curvatures.

The appearance of additional extremal curvatures of the surface results in more complex behavior of the system in this area of the configuration space (the appearance of different nonlinear effects) and is a condition for transition to other, more complex vibration modes and for an energy flow between the coordinates. This energy flow results in the appearance of additional frequency components in the frequency analysis of projections of an image point's trajectories on a coordinate frame (usually the Fourier method or wavelet analysis is used). The surface topology determines the frequency and energy spectra, which are well reproduced by the wavelet spectrum taking into account local singularities in the system dynamics.

We now consider as an example an analysis of the stability of the equilibrium of a double pendulum with bodies M_1 and M_2 connected to bars (with masses m_1 and m_2) considered as material points [7]. The bar masses, resistance of air, and friction in horizontal cylindrical supports are neglected; when the pendulum is in the upper vertical position, spiral springs with the rigidities C_1 and C_2 fixed in hinges are not deformed. The system restraints are ideal, stationary, and holonomic and the active forces applied to the system are conservative ones. The positions of the pendulums are set by the angles X_1 and X_2. The potential energy P is the sum of the potential energy of the springs P_1 and the potential energy of gravity forces P_2

$$P_1 = \frac{1}{2}C_1X_1^2 + \frac{1}{2}C_2(X_1 - X_2)^2, \quad P_2 = (m_1 + m_2)gl_1(1+\cos X_1) + m_2gl_2(1+\cos X_2).$$

The nonlinear properties of the system are described in the potential energy by the P_2 term. The dominance of one or the other component of the potential energy determines the degree of the system's nonlinearity.

In [8], the area of stability was determined for this system after its linearization using the Sylvester criterion. For the given values of the pendulum masses m_1 and m_2 and their lengths l_1 and l_2, the spring rigidities C_1 and C_2 are determined in such a way as to ensure that the pendulum equilibrium in the upper position is stable.

If the rigidity of the springs increases, the upper position becomes stable. In this approach, which was used in [8], nonlinear properties and their effect on the stability of the system actually were not considered. An analysis of the system with nonlinear effects that involves plotting energy surfaces yields a more in-depth understanding of the specific features of the system.

In determining the energy surface, the considered configuration space includes the upper and lower positions of the pendulums and their combination. The following theorems may be used for analyzing the stability of the different positions of the pendulums on the basis of the energy surfaces [8]: the Lagrange–Dirichlet theorem, stating that if potential energy is minimal in an equilibrium state, the equilibrium state is stable, and the inverse Lyapunov theorem, stating that if potential energy is not minimal in an equilibrium state, the equilibrium state is not stable.

In absence of the springs C_1 and C_2, the system has five elliptic and four hyperbolic points. The elliptic point (the surface curvature at this point is negative) at the origin of the frame corresponds to an unstable upper vertical position of the double pendulum ($P = P_{max}$). The other elliptic points (the surface curvature is positive) are characterized by the lower position of both pendulums (stable positions). Between the elliptic points, hyperbolic points ($P \neq 0$) are located; in this case, one pendulum is in the lower position and the other is in the upper position ($0 < P < P_{max}$). These points are stable for some relative motions of the pendulums and unstable for others (the surface curvatures at these points have different signs).

If the rigidity of the springs increases, the energy surface experiences a transformation. At some value of the spring rigidity, the elliptic point with $X_1 = X_2 = 0$ ($P = P_0$) becomes a hyperbolic one; i.e., one of its directions is stable and the other is unstable. There are also two elliptic points with a smaller potential energy ($P_i < P_0$), which determine the stable positions of the pendulums. In this case, the stability of the pendulums in the upper vertical positions is attained if the pendulums oscillate with opposite phases.

For this degree of transformation of the energy surface, the regime depends on the initial conditions. If $P < P_0$, vibrations will develop around either one or the other stable elliptic point. If $P > P_0$, there will be a regime in the area of the elliptic points where hopping occurs between the areas across the area of a hyperbolic point. Systems with such properties, where a particle moves in a potential with two minima, are called bistable systems. Such a model is applicable to many physical and biological phenomena. In [9], a description is contained of an experiment with a laser that may emit two modes with different polarizations corresponding to two stable states. Due to noise, the laser generation switches between emission in these modes in an irregular way. The frequency of hops between the potential wells depends on the noise intensity. If the noise is weak, the hops are rare; if the noise intensity grows, they become more frequent. If the noise is very strong, the presence of stability points is not seen. The implementation of different modes depends on the relation between the noise intensity and the level of the energy barrier (a hyperbolic point).

If the spring rigidity increases further, the in-phase motion of the pendulums also becomes stable. The system becomes a quasi-linear one with a single stable elliptic point.

To illustrate the character of the transformation of an energy surface, we will consider the problem of stabilization of an inverted double pendulum by high-frequency vibrations applied to the suspension point [10]. If the suspension point of the pendulum vibrates in the vertical direction with a high frequency ω and an amplitude a, the force acting on the pendulum is $F = -mla^2\cos(\omega t)\sin(X) = F_0\cos(\omega t)$, $F_0 = -mla^2\sin(X)$. Applying the averaging method, one can represent this force as an addition of the corresponding energy component to the potential energy of the system. To do this, we introduce an effective potential [11] $P = P_0 + \Delta P_\omega = P_0 + (1/2m\omega^2)\overline{F}_0^2$, where the bar over F_0 means averaging over fast vibrations and P_0 is the potential energy of the system without the vibration effect. From this expression, we find the value of the energy addition $\Delta P_\omega = mgl\delta\sin^2(X)/4$, where $\delta = (a/l)^2(\omega/\omega_0)^2$.

For a double pendulum, high-frequency vibration affects both coordinates. In this case, the additional energy term is $\Delta P_\omega = (m_1 + m_2)gl_1\delta_1\sin^2(X_1)/4 + m_2gl_2\delta_2\sin^2(X_2)/4$. If the links of a double pendulum have the same characteristics, $\delta_1 = \delta_2 = \delta$. Figure 2 shows the energy surface of the system for zero rigidity of the strings taking into account the additional energy ΔP_ω.

If high-frequency vibration is applied to the suspension axis of a pendulum, the transformation of the energy surface results in the appearance of a surface depression in the vicinity of the unstable elliptic point, which becomes a stable one. This point is surrounded by four additional unstable elliptic points and four hyperbolic points.

If, by changing the rigidity of the springs in the pendulum hinges, a bistable system may be created, introduction of high-frequency vibration applied to the suspension point of the pendulum may result (in addition to the two other areas) in the appearance of a new area of stability, which is complicated by the presence of additional points.

We now present another example. Let us consider a system of Rössler equations [12] $\dot{R}_1 = -(R_2 + R_3)$, $\dot{R}_2 = R_1 + aR_2$, $\dot{R}_3 = a + R_1R_3 - cR_3$, $R_i(0) = R_{0i}$, $i = 1, 2, 3$, where R_1, R_2, and R_3 are coordinates and a and c are some coefficients in the equations.

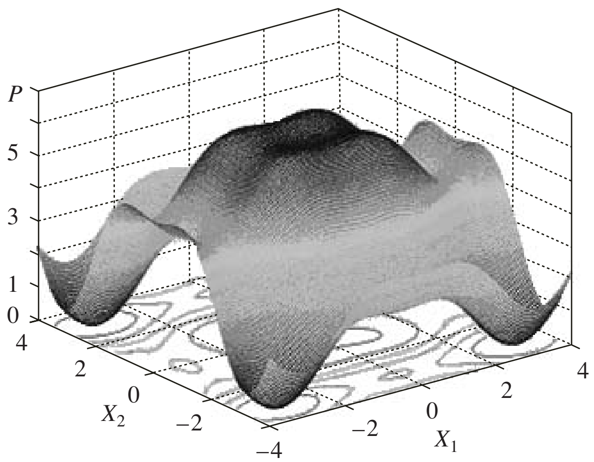

Fig. 2. Potential energy surfaces under the effect of high-frequency vibration on the suspension axis of a double pendulum.

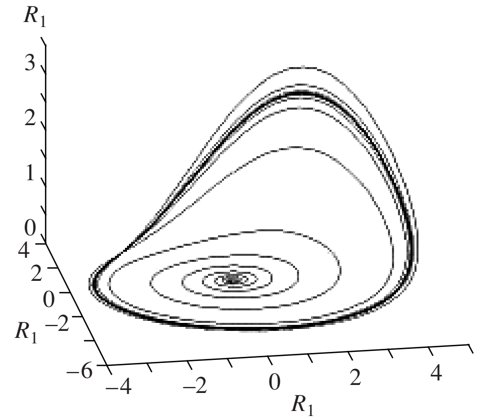

Fig. 3.

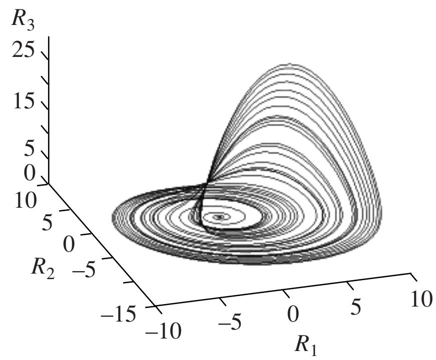

Fig. 4.

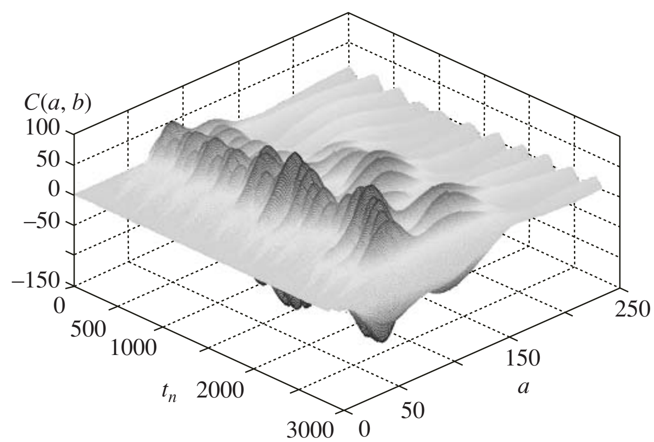

Fig. 5. Wavelet spectrum of a signal $C(a, b)$: t_n is time (the sequential number of an element in a time series), a determines the frequency, and $C(a, b)_{max}$ characterizes the frequency and energy spectra.

This model is introduced as an illustration of the complex interactions between subsystems and the conditions under which dynamic regimes change. This model has three degrees of freedom and shows the richness of possible dynamic modes that depend on the characteristic features of the potential energy surface (the potential function).

An analysis of the properties of the potential function for this system shows that there are two characteristic points $(a, 0, 0)$ and $(-a + 2c, 0, 2)$ on the potential energy surface. The mutual spatial position of these points and their types determine the character of the geodetic lines on the potential energy surface and, consequently, the dynamic characteristics of the system. In this case, both points belong to the parabolic type (the Gaussian curvatures of the surface at the characteristic points are zero) with two coordinates and an neutral equilibrium. The third coordinate is unstable for the first point and stable for the other point. The spatial location and character of these points determine the trajectories of motion of the system (see Fig. 3 for $a = 0$ and 2 and $c = 2$ and 1 and Fig. 4 for $a = 0$ and 3 and $c = 4$ and 6). The considered examples differ by the values of the coefficients a and c, which determine the spatial distribution of the characteristic points.

Figure 5 shows a wavelet spectrum for the regime of motion presented in Fig. 4. This regime differs from the previous one by the number of frequencies; the first regime is characterized by a single stable frequency and this regime has two frequencies that are variable.

The Lorentz system, which was used as a simplified model of atmospheric convection, is widely known. Lorentz found that the solution of the system of equations depends on the initial conditions; this phenomenon may be explained by the properties of geodetic lines [13]. Lorentz reduced the complex system of equations for the flow function and the temperature field to a system of three equations [12]: $\dot{X} = \sigma(Y - X)$, $\dot{Y} =$

$-XZ + rX - Y$, $\dot{Z} = XY - bZ$. This system has the characteristic points (0, 0, 0), ($\sqrt{b(r-1)}$, $\sqrt{b(r-1)}$, $r-1$), and ($-\sqrt{b(r-1)}$, $-\sqrt{b(r-1)}$, $r-1$), which determine the character of the trajectory of the image point's motion in the configuration space. Changes to coefficients b and r result in changes in the trajectories of motion of the system (usually, when the Lorentz system is represented, only one variant of the widely known motion regime is shown).

The examples presented show the efficiency of the developed method for qualitative and quantitative analysis of the specific features of the dynamic behavior of nonlinear systems. Based on the analysis of the specific features of the potential energy surface, this method enables one to obtain a clear geometric generalization of the dynamic properties of systems and facilitates comprehension of the numerical results obtained in mathematical models.

The analysis of these specific features of surfaces is necessary for revealing nonlinear effects in systems and is a basis for selecting the dimensionality of a nonlinear system and its identification.

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